Bayes decision rule

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Costs

Classification

- supervised learning
- $y \in \{1, 2, ...C\}$ takes finite discrete set of values

- λ_{yf} is the cost of predicting true class y with forecasted class f.
- Examples with costs: diagnosis prediction, fraud detection, spam filtering, intrusion detection.

Costs

- Matrix of outcomes:

	$f=1$	$f=2$	$\cdots$	$f=C$
$y=1$	λ_{11}	λ_{12}	$\cdots$	λ_{1C}
$y=2$	λ_{21}	λ_{22}	$\cdots$	λ_{2C}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$y=C$	λ_{C1}	λ_{C2}	$\cdots$	λ_{CC}

- Expected cost of solution $\widehat{y}(x)=f$:

$$\mathcal{L}(f)=\sum_{y}p(y|x)\lambda_{yf}$$

Decision rule

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Bayes minimum risk decision rule

Assign class, yielding minimum expected cost:

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$$\widehat{y}(x) = \arg\min_{f} \mathcal{L}(f) \tag{1}$$

- This rule minimizes expected cost among all rules (if $p(y|x)$ are correct).

Simplifications

- $\lambda_{yf} \equiv \lambda_y \mathbb{I}[y \neq f]$: constant within class cost of misclassification.

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$y=1$	0	λ_1	$\cdots$	λ_1
$y=2$	λ_2	0	$\cdots$	λ_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$y=C$	λ_C	λ_C	$\cdots$	0

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- Expected cost of solution $\widehat{y}(x)=f$: $\mathcal{L}(f)=\sum_y p(y|x)\lambda_y \mathbb{I}[f \neq y]$

Equal misclassification costs

- Then cost of prediction equals:

$$\mathcal{L}(f) = \sum_y p(y|x)\lambda_y \mathbb{I}[f \neq y] = \overbrace{\sum_y p(y|x)\lambda_y}^{const(f)} - p(f|x)\lambda_f$$

- So (1) becomes:

$$\widehat{y}(x) = \arg\min_f \mathcal{L}(f) = \arg\max_f \lambda_f p(f|x) \tag{2}$$

- Suppose further $\lambda_y \equiv \lambda\,\forall y$, then

$$\widehat{y}(x) = \arg\max_f p(f|x)$$

- This is termed **maximum posterior probability rule** or **Bayes minimum error rule** because it yields minimum probability of misclassification among all decision rules (given that $p(f|x)$ is correct)

Equal misclassification costs

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Equal misclassification costs

- This rule minimizes expected error rate.
  - if $p(y|x)$ are known
- If x and y are independent, then (2) reduces to

$$\widehat{y}(x) = \arg\max_f p(f|x) = \arg\max_f p(f)$$

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Gaussian classifier

- In Gaussian classifier
$$p(x|y)=\frac{1}{(2\pi)^{D/2}|\Sigma_y|^{1/2}}exp\left\{-\frac{1}{2}(x-\mu_y)^T\Sigma_y^{-1}(x-\mu_y)\right\}$$

Gaussian classifier

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- It follows that

$$\begin{aligned}\log p(y|x) &= \log p(x|y)+\log p(y)-\log p(x)\\ &= -\frac{1}{2}(x-\mu_y^T)\Sigma_y^{-1}(x-\mu_y)-\frac{1}{2}\log|\Sigma_y|\\ &\quad -\frac{D}{2}\log(2\pi)+\log p(y)-\log p(x)\end{aligned}$$

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- Removing common additive terms, we obtain discriminant functions:
$$g_y(x)=\log p(y)-\frac{1}{2}\log|\Sigma_y|-\frac{1}{2}(x-\mu_y)^T\Sigma_y^{-1}(x-\mu_y) \quad (3)$$

Practical application

- In practice we replace theoretical terms μ_y, Σ_y with their sample estimates $\widehat{\mu}_y$, $\widehat{\Sigma}_y$.
- $\widehat{p}(y) = \frac{N_y}{N}$.

$$g_y(x) = \log \widehat{p}(y) - \frac{1}{2}\log|\widehat{\Sigma}_y| - \frac{1}{2}(x - \widehat{\mu}_y)^T \widehat{\Sigma}_y^{-1}(x - \widehat{\mu}_y)$$

- Analysis:
    - depends on normality assumptions (in particular - on unimodality)
    - needs to specify:
        - CD parameters to estimate $\widehat{\mu}_y$, $y = 1, 2, ...C$.
        - $CD(D+1)/2$ parameters to estimate $\widehat{\Sigma}_y$, $j = 1, 2, ...C$.

Simplifying assumptions

- $CD(D+3)/2$ may be too large for multidimensional tasks with small training sets.
- Simplifying assumptions:
  - **Naive Bayes**: assume that $\Sigma_1, \Sigma_2, ...\Sigma_C$ are diagonal.
  - **Project data onto a subspace**: for example on first few principal components.
  - **Proportional covariance matrices**: assume that $\Sigma_1 = \alpha_1\Sigma$, $\Sigma_2 = \alpha_2\Sigma$, ... $\Sigma_C = \alpha_C\Sigma$.
  - **Fisher's linear discriminant analysis**: assume that $\Sigma_1 = \Sigma_2 = ... = \Sigma_C$.

QDA vs. LDA

Gaussian classifier is called:

- *Quadratic discriminant analysis* (QDA) when $\Sigma_1, \Sigma_2, ...\Sigma_C$ are arbitrary.
  - class boundaries are quadratic[1]
- *Linear discriminant analysis* (LDA) when $\Sigma_1 = \Sigma_2 = ... = \Sigma_C$
  - class boundaries are linear[2]

[1] prove this
[2] prove this

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High dimensional problem

$p(x^1, x^2, ...x^D) = p(x^1)p(x^2|x^1)...p(x^D|x^1, x^2, ...x^{D-1})$

Problem: exponential to D number of observations needed for estimation.

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Individual features are independent: $p(x) = p(x^1)p(x^2)...p(x^D)$

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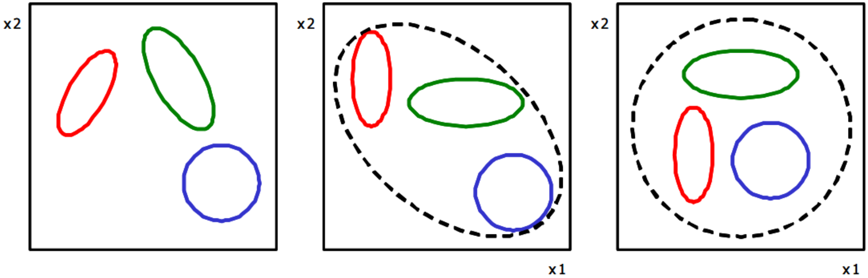
Naive Bayes assumption in classification

Individual features are **class conditionally** independent:
$p(x|y) = p(x^1|y)p(x^2|y)...p(x^D|y)$

Under Naive Bayes assumption max-posterior probability rule becomes:

$$\widehat{y}(x) = \arg\max_y p(y)p(x^1|y)p(x^2|y)...p(x^D|y)$$

Conditional independence visualization



$p(x|\omega_i) \neq \prod_{d=1}^{D} p(x_d|\omega_i)$

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