# Efficient approximability of the Euclidean Capacitated Vehicle Routing Problem with Time Windows and Splittable Demand

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# Capacitated Vehicle Routing Problem (CVRP) :: Motivation

## Practical applications

See, e.g. [Vehicle Routing. Problems, Methods and Applications (Toth, Vigo. 2014)].


## The spheres of applications

- Oil, gas and fuel transportation
- Retail applications
- Waste collection and management
- Mail and Small package delivery
- Food distribution

# Capacitated Vehicle Routing Problem (CVRP) :: Problem statement

## CVRP

**Instance:**

- a complete edge-weighted digraph $G = (X \cup \{y\}, E, w)$
- each customer $x_i$ has a unit demand
- each vehicle has the same capacity $q$
- any feasible route has the form

$$R = y, x_{i_1}, \ldots, x_{i_s}, y, \text{ where } s \leq q$$

and the cost $w(R) = w(y, x_{i_1}) + w(x_{i_1}, x_{i_2}) + \ldots + w(x_{i_s}, y)$

**Goal:** to find a collection $S = \{R_1, \ldots, R_b\}$ of feasible routes visiting each customer exactly once and having the minimum total cost $w(S) = \sum_{j=1}^{b} w(R_j)$

# Capacitated Vehicle Routing Problem with Time Windows and splittable demand (CVRPTW-SD) :: Problem statement

## CVRPTW-SD

**Instance:**

- a complete edge-weighted digraph $G = (X \cup \{y\}, E, w)$
- a set $T = \{t_1, \ldots, t_p\}$ of mutually disjoint time windows, wlog. $t_i \preceq t_j$ for any $i \leq j$
- each vehicle has the same capacity $q$
- each customer $x_i$ has a splittable demand $d(x_i)$ and should be visited at time window $T(x_i) \in T$

# Capacitated Vehicle Routing Problem with Time Windows with splittable demand(CVRPTW-SD) :: Problem statement

## CVRPTW-SD

**Instance (ctd):**

- A *feasible route* is an ordered pair $\mathcal{R}_j = (R_j, D_j)$, where $R_j = y, x_{i_1}, \ldots, x_{i_s}, y$ is a closed tour in the graph $G$ and the $n$-tuple $D_j = (d_{1j}, \ldots, d_{nj})$ fulfills the time windows

$$T(x_{i_l}) \preceq T(x_{i_{l+1}}), \quad (1 \leq l < s)$$

and capacity

$$\begin{aligned} 1 \leq d_{i_l j} \leq d_{i_l}, & \qquad (1 \leq l \leq s) \\ d_{ij} = 0, & \qquad i \notin \{i_1, \ldots, i_s\} \\ \sum_{i=1}^{n} d_{ij} \leq q & \end{aligned}$$

constraints, where $d_{ij}$ is a part of the $i$ customer covered by $R_j$.
The transportation cost is $w(R_j) = w(y, x_{i_1}) + w(x_{i_1}, x_{i_2}) + \ldots + w(x_{i_s}, y)$

# Capacitated Vehicle Routing Problem with Time Windows with splittable demand(CVRPTW-SD) :: Problem statement

## CVRPTW-SD

**Goal:** to find, for some $m \geq 1$, a minimum cost multi-cover $\mathcal{U} = (\mathcal{R}_1, \ldots, \mathcal{R}_m)$ of the graph $G$, satisfying the total customer demand, i.e.

$$\sum_{j=1}^{m} d_{ij} = d_i, \quad (1 \leq i \leq n).$$

# CVRP and CVRPTW-SD Related work

## CVRP

- (G.Dantzig and J.Ramser, 1959) Introduced the CVRP problem
- (M.Haimovich and A.Rinooy Kan, 1985) First PTAS algorithm for $q = o(\log\log n)$
- (T.Asano et.al, 1996) Improving PTAS algorithm for $q = O(\log n / \log\log n)$
- (C.Adamaszek, and A.Czumaj, and A.Lingas, 2009) PTAS for k-tour cover problem on the plane for moderately large values of k
- (A.Das and C.Mathieu, 2015) Quasi-Polynomial Time Approximation Scheme (QPTAS) for the Euclidean plane with time complexity $n^{\log n^{O(1/\epsilon)}}$
- (M.Khachay and R.Dubinin, 2016) First EPTAS for the CVRP in the Euclidean space of an arbitrary dimension $d > 1$

# CVRP and CVRPTW-SD Related work

## CVRPTW

- (P.Toth and D.Vigo, 2014) Many efficient branch-and-cut methods and numerous heuristics
- (L.Song and H.Huang and H.Du, 2016) Extending QPTAS for CVRP to the case of finite number of non-intersecting time-windows with time complexity $n^{\log^{O(1/\epsilon)} n}$
- (M.Khachay, and Y. Ogorodnikov, 2018) Efficient PTAS for the Euclidean CVRP with Time Windows
- (M.Khachay, and Y. Ogorodnikov, 2018) Improved Polynomial Time Approximation Scheme for Capacitated Vehicle Routing Problem with Time Windows

# Approximation schemes :: possible time complexities

(i) QPTAS: $O\left(n^{poly(\log n)^{O(1/\varepsilon)}}\right)$

(ii) PTAS: $O\left(n^{\exp(\frac{1}{\varepsilon^3})}\right)$

(iii) EPTAS: $O\left(\exp^{\exp^{\frac{1}{\varepsilon^4}}} \cdot n^3\right)$, $O\left(n^3 + \exp^{1/\varepsilon^4}\right)$

(iv) FPTAS: $O\left(\left(\frac{1}{\varepsilon}\right)^{10} \cdot n^{15}\right) = poly\left(n, \frac{1}{\varepsilon}\right)$

# CVRPTW-SD:: Our approximation scheme

## The goal and main idea

**Our goal:** to develop a PTAS algorithm for the CVRPTW-SD on the Euclidean plane with $w(x_i, x_j) = \|x_i - x_j\|_2$

**Our approach:** combines

- the approach proposed by C.Adamaszek et al for the Euclidean CVRP
- QPTAS proposed by L.Song at el. for the Euclidean CVRPTW

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## The scheme :: main stages

- Preprocessing.
- Rounding.
- Instance decomposition onto white and gray subinstances.
- Blackboxing: applying Song's QPTAS for any white subinstance and the ITP heuristic for the grey ones.

# Approximation scheme :: preliminaries

**Lemma 1**

For any instance of the $CVRPTW$-$SD$, such that $r_1 \geq \ldots \geq r_n$, $r_i = \min\{w(y, x_i) \colon y \in Y\}$, the following equation

$$\text{OPT} \geq \max\left\{\text{TSP}^*(X \cup \{y\}), 2r_1, \frac{2}{q}\sum_{i=1}^{n} d_i r_i\right\}$$

is valid.

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is valid.

Introduce $\rho$ and $N$

$$\rho = \frac{r_1 \varepsilon}{N}, \text{ where } N = \sum_{i=1}^{n} \left\lceil \frac{d_i}{q} \right\rceil,$$

**Lemma 2**

Demand of all customers, for which $r_i \leq \rho$, can be serviced by routes of at most $\varepsilon \cdot \mathrm{OPT}$ total cost.

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**Lemma 3**

$$w(S_{ITP}) \leq 2 \cdot \left(\frac{2}{q}\sum_{i=1}^{n} d_i r_i\right) + p w(H) \leq 2 \cdot \left(\frac{2}{q}\sum_{i=1}^{n} d_i r_i\right) + p\beta \cdot \mathrm{TSP}^*(X).$$

# Approximation scheme :: preliminaries

**Trivial and non-trivial routes**

We call a feasible route $\mathcal{R}$ *non-trivial*, if it visits at least two distinct customers, i.e. $|X(\mathcal{R})| > 1$. Otherwise, the route is called *trivial*.

# Approximation scheme :: preliminaries

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**Lemma 4 [Adamaszek, etc., 2009]**

For any instance of the CVRP, there exists an optimum solution $\mathcal{S} = \{\mathcal{R}_1, \ldots, \mathcal{R}_m\}$, such that, among its $m$ routes, at most $|X|$ are non-trivial.

# Approximation scheme :: preliminaries

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### Lemma 5

For any instance of the CVRPTW-SD, there exists an optimum solution $\mathcal{S} = \{\mathcal{R}_1, \ldots, \mathcal{R}_m\}$ having at most $T$ non-trivial routes, where $T$ is the number of slots.

# Approximation scheme

## Preprocessing

Relabel the customers in the order $r_1 \geq r_2 \geq \ldots \geq r_n$, where $r_i = w(y, x_i)$. Then, given an $\varepsilon > 0$, we set a tolerance threshold

$$\rho = \frac{r_1 \varepsilon}{N}, \text{ where } N = \sum_{i=1}^{n} \left\lceil \frac{d_i}{q} \right\rceil,$$

and exclude all the customers $x_i$, for which $r_i \leq \rho$.

# Approximation scheme

## Rounding

We introduce the accuracy dependent grid induced by the circles centered at the depot $y$ of radii

$$\rho_i = \rho \left(1 + \frac{\varepsilon}{q}\right)^i, \; 0 \leq i \leq \lceil \log_{1+\frac{\varepsilon}{q}} N/\varepsilon \rceil$$

Divide circles into sectors of $\lceil \frac{2\pi q}{\varepsilon} \rceil$ angle, and move clients to the nearest location. We call *locations* the obtained intersection points between rays and circles. To any location, we assign $p$ *slots*.

# Approximation scheme

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We introduce the accuracy dependent grid induced by the circles centered at the depot $y$ of radii

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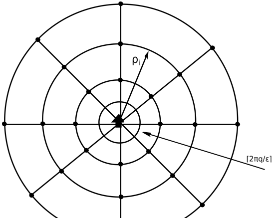


The total number of slots: $\Theta\left(p \cdot \left(\frac{q}{\varepsilon}\right)^2 \log \frac{N}{\varepsilon}\right)$

# Approximation scheme

**Lemma 6**

The proposed reduction changes the cost of any solution by at most $\varepsilon \cdot \mathrm{OPT}$.

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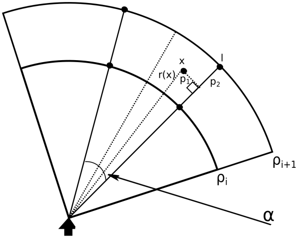


## Sketch of the proof

It is easy to verify, that

$\|x - l\|_2 \leq p_1 + p_2 \leq r(x)\alpha/2 + (\rho_{i+1} - \rho_i)/2.$

Therefore, $\|x - l\|_2 \leq r(x)\frac{\varepsilon}{q}$. It can be seen, that the total change of routes does not exceed

$$\varepsilon \cdot \frac{2}{q} \sum_{i=1}^{n} d_i r_i \leq \varepsilon \cdot \text{OPT},$$

by Lemma 1.

# Instance decomposition

## Instance partition to rings

(i) Partition the enclosing disk (of radius $r_1$) to *rings*, such that each ring (except maybe the most inner one) consists of $k = \lceil \log_{1+\frac{\varepsilon}{q}} \frac{5}{\varepsilon} \rceil$ consecutive circles

(ii) For the positive integer $a = \lceil (20p\beta + 4)/\varepsilon \rceil$ and some number $b \in \{0, \ldots, a-1\}$, starting from the outer one, color all the rings obtained in white and gray, such that the ring $K_i$ is colored gray, if $i \equiv b \pmod{a}$.

# Instance decomposition

## Ring width :: lower bound

For any ring $K$, $r_{out} = r_{in}(1+\varepsilon/q)^k$, where $k = \lceil \log_{1+\frac{\varepsilon}{q}} \frac{5}{\varepsilon} \rceil$.
Then, for its width $W(K)$,

$$W(K) = r_{in}\left(\left(1+\frac{\varepsilon}{q}\right)^k - 1\right) \geq r_{in}\left(\left(1+\frac{\varepsilon}{q}\right)^{\log_{1+\frac{\varepsilon}{q}} \frac{5}{\varepsilon}} - 1\right)$$
$$= r_{in}\left(\frac{5}{\varepsilon} - 1\right) \geq 2r_{in}\frac{2}{\varepsilon},$$

i.e.

$$2r_{in} \leq \frac{\varepsilon}{2} \cdot W(K)$$

# Instance decomposition

**White families**

By $\mathfrak{F}_1, \ldots, \mathfrak{F}_\alpha$ and $\mathrm{OPT}(\mathfrak{F}_i)$ we denote the maximal (by inclusion) families of white consecutive rings and the optimum value of the CVRPTW-SD subinstance induced by slots located in rings of the family $\mathfrak{F}_i$, respectively.

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## Lemma 7

For any white-gray coloring of rings obtained by the following rules:

(i) any monochromatic pair of the adjacent rings is white,

(ii) the outer ring is white as well,

the following equation

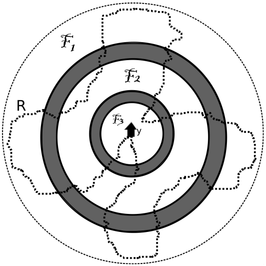
$$\sum_{i=1}^{\alpha} \mathrm{OPT}(\mathfrak{F}_i) \leq \left(1 + \frac{\varepsilon}{2}\right) \mathrm{OPT}$$

is valid.

# Instance decomposition

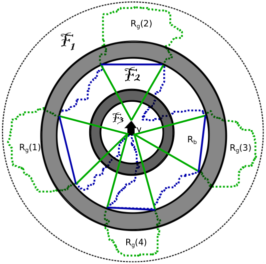
*Inspired by paper* **Baker, B.S.: Approximation algorithms for NP-complete problems on planar graphs. Journal of the ACM 41(1), 153180 (1994)**

Let $\mathcal{U} = \{\mathcal{R}_1, \dots \mathcal{R}_m\}$ be an arbitrary optimum solution of the initial rounded instance of the CVRPTW-SD.



# Instance decomposition

Take any route $\mathcal{R}$ from the solution $\mathcal{U}$ and shortcut onto $\mathcal{R}_g(1), \mathcal{R}_g(2), \dots, \mathcal{R}_g(l)$ subroutes




## Sketch of the proof

(i) Transformation results by the one step for the single route is increasing of the transportation cost by at most $4 \cdot r_{in} \cdot l \leq 2l \cdot \varepsilon/2 \cdot W(K) \leq \varepsilon/2 \cdot w(\mathcal{R} \cap K)$

(ii) The total cost increasing caused by such a transformation for the single route $\mathcal{R}$ does not exceed $\frac{\varepsilon}{2} \cdot \sum_{j=1}^{\alpha} w(\mathcal{R} \cap K_j)$,

(iii) The total cost of the obtained routes is at most $w(\mathcal{U}) + \frac{\varepsilon}{2} \sum_{i=1}^{m} \sum_{j=1}^{\alpha} w(\mathcal{R}_i \cap K_j) \leq (1 + \varepsilon/2) w(\mathcal{U})$.

# Instance decomposition

**Lemma 8**

Let $K_1, \ldots, K_\alpha$ be the gray rings. Then,

$$\sum_{i=1}^{\alpha} \mathrm{TSP}^*(K_i) \leq (1 + \pi\varepsilon)\,\mathrm{TSP}^*.$$

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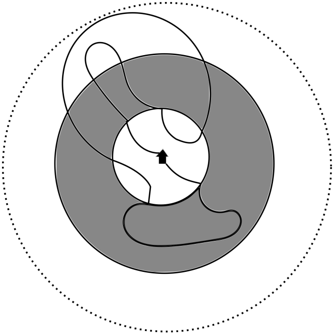

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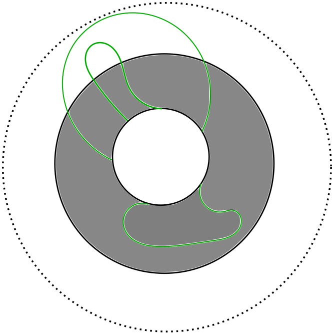

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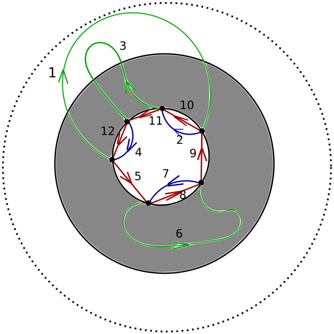

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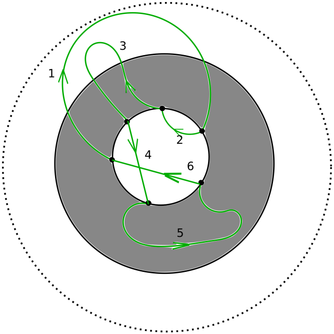

# Instance decomposition

## Sketch of the proof of Lemma 8

(i) The upper bound for the cost $w(H(K))$ of the constructed cycle $H(K)$

$$\begin{aligned} w(H(K)) \le w(E(K)) &\le w_{ext}(K) + 4\pi \cdot r_{in} \\ &\le w_{ext}(K) + \pi\varepsilon \cdot W(K) \le w_{ext}(K) + \pi\varepsilon \cdot w(H \cap K), \end{aligned}$$

(ii) The final bound for the all gray rings

$$\sum_{i=1}^{\alpha} \mathrm{TSP}^*(K_i) \le \sum_{i=1}^{\alpha} w(H(K_i)) \le (1 + \pi\varepsilon)\,\mathrm{TSP}^*.$$

# Instance decomposition

**N.B.**

Lemma 8 is valid for an arbitrary white-gray coloring. In particular, for the case, when each family $\mathfrak{F}$ contains only a single ring

**Lemma 9**

Let $\mathrm{TSP}^*(K_i)$ be the optimum value for the Euclidean TSP instance enclosed in the ring $K_i$. Then, the following equation holds:

$$\sum_{i=1} \mathrm{TSP}^*(K_i) \leq 10 \cdot \mathrm{TSP}^*.$$

# Instance decomposition

## Lemma 9

Let $\mathrm{TSP}^*(K_i)$ be the optimum value for the Euclidean TSP instance enclosed in the ring $K_i$. Then, the following equation holds:

$$\sum_{i=1} \mathrm{TSP}^*(K_i) \le 10 \cdot \mathrm{TSP}^*.$$

## Sketch of the proof of Lemma 9

We obtain the following equation for two alternative colorings

$$\sum_{i=1}^{k} \mathrm{TSP}^*(K_i) = \sum_{i \equiv 0 \pmod 2} \mathrm{TSP}^*(K_i) + \sum_{i \equiv 1 \pmod 2} \mathrm{TSP}^*(K_i)$$

$$\le 2\,(1+\pi\varepsilon)\,\mathrm{TSP}^* + \mathrm{TSP}^*(K_1) \le 2\,(1+\pi\varepsilon)\,\mathrm{TSP}^* + \mathrm{TSP}^* \le 10 \cdot \mathrm{TSP}^*,$$

since $\varepsilon < 1$.

# Instance decomposition

**Lemma 10. Total cost of the ITP solutions**

There exists a number $b \in \{1, \ldots, a\}$, such that the total cost of all ITP solutions for the subinstances enclosed in the gray rings is at most $\frac{\varepsilon}{2} \cdot \mathrm{OPT}$.

# Instance decomposition

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There exists a number $b \in \{1, \ldots, a\}$, such that the total cost of all ITP solutions for the subinstances enclosed in the gray rings is at most $\frac{\varepsilon}{2} \cdot \mathrm{OPT}$.

**Sketch of the proof of Lemma 10**

Indeed,

$$w(S_{\mathrm{ITP}}(K)) \leq 2 \cdot \frac{2}{q} \sum_{x \in X_{slots}(K)} d(x) r(x) + p\beta \cdot \mathrm{TSP}^*(K)$$

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**Sketch of the proof of Lemma 10**

Therefore, by Lemmas 2, 9, 1

$$\sum_{b=0}^{a-1} \sum_{i \equiv b \pmod{a}} w(S_{\mathrm{ITP}}(K_i)) \leq 2 \cdot \frac{2}{q} \sum_{x \in \mathfrak{S} K} d(x) r(x) + p\beta \cdot \sum_{i=1}^{k} \mathrm{TSP}^*(K_i)$$
$$\leq 2 \cdot \mathrm{OPT} + 10p\beta \cdot \mathrm{TSP}^* \leq (2 + 10p\beta)\mathrm{OPT}.$$

Hence, there exists $b$, such that

$$\sum_{i \equiv b \pmod{a}} w(S_{\mathrm{ITP}}(K_i)) \leq \frac{2 + 10p\beta}{a} \mathrm{OPT} \leq \frac{\varepsilon}{2} \mathrm{OPT}.$$

# Main Result and blackboxing

From Lemmas 7 and 10 we have

**Theorem 1**

For any $\varepsilon \in (0, 1)$, the proposed decomposition provides $(1 + \varepsilon)$-approximate solution for the initial rounded CVRPTW-SD instance.

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**Theorem 1**

For any $\varepsilon \in (0, 1)$, the proposed decomposition provides $(1 + \varepsilon)$-approximate solution for the initial rounded CVRPTW-SD instance.

**Blackboxing**

(i) For the white subinstances we will use Song's QPTAS [Song, 2016]

(ii) For the gray subinstances we will use the ITP heuristic

# Time complexity bounds

**Theorem 2**

Time complexity of the proposed scheme is

$$O\left(I \cdot \mathcal{K}(p,q,\varepsilon) + n \log n\right),$$

where

$$I = O\left(\frac{\varepsilon}{p} \frac{\log \frac{N}{\varepsilon}}{\log \frac{1}{\varepsilon}}\right), \text{ here } N = \sum_{i=1}^{n} \left\lceil \frac{d_i}{q} \right\rceil$$

and

$$\mathcal{K}(p,q,\varepsilon) = \left(\sigma_w^2 q\right)^{\left(\log(\sigma_w^2 q)\right)^{O(1/\varepsilon)}} + (\sigma_g^2 q)^3,$$

and

$$\sigma_w = O\left(\frac{(pq)^2}{\varepsilon^3} \cdot \log \frac{1}{\varepsilon}\right),$$

and

$$\sigma_g = O\left(\frac{pq^2}{\varepsilon^2} \cdot \log \frac{1}{\varepsilon}\right)$$

# Time complexity bounds

**Corollary 1**

For any fixed $\varepsilon \in (0, 1)$, the running time of the proposed scheme does not exceed $O(n \log N)$, if $p = \Omega(1)$, $q = \Omega(1)$, and

$$\max\{p, q\} \leq 2^{\log^{\delta} n}$$

for some $\delta = O(\varepsilon)$.

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**Corollary 2**

For any fixed $p$ and $q$ the proposed scheme is EPTAS with time complexity $O\left(\left(\frac{1}{\varepsilon^8}\right)^{\left(\log \frac{1}{\varepsilon}\right)^{O(1/\varepsilon)}} \cdot \log N + n \log n\right)$.

# Conclusion and future work

## Conclusion

(i) Perhaps, first approximation scheme for CVRP with time windows and splittable demand

(ii) For any fixed $\varepsilon \in (0, 1)$ and the total customer demand $D$, the scheme finds a $(1 + \varepsilon)$-approximate solution of the problem in time $O(n \log D)$ any time, when $\max\{p, q\} \leq 2^{\log^{\delta} n}$ for some $\delta = O(\varepsilon)$

(iii) for any fixed capacity $q$ and the number $p$ of time windows, the proposed scheme is EPTAS

## Future work

(i) Extension to an arbitrary finite dimension of Euclidean space

(ii) Extension for the non-splittable demand

*Thank you for your attention!*